A Formalization of Prokhorov's Theorem in Isabelle/HOL

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Today's Talk

A Formalization of the Lévy-Prokhorov Metric in Isabelle/HOL, ITP2024.

The Lévy-Prokhorov Metric
= A metric between finite measures on a metric space.

- Lévy-Prokhorov metric
- Prokhorov's theorem
- The space of all finite measures is a Polish and standard Borel space

Today's Talk

Prokhorov's Theorem
(I did not talk the detail in the presentation at ITP2024)

What is Prokhorov's theorem?

Measure

A measure μ on X

$$\boxed{\mu : \Sigma_X \to [0, \infty]}$$

X : set, $\Sigma_X \subseteq 2^X$: σ-algebra on X.

Intuitively, $\mu(A) =$ the *size* of A
(finite measure $\iff \mu(X) < \infty$)

Ex.

The Lebesgue measure ν on $\mathbb{R}^n$	$\nu\left((a_i, b_i]^n\right) = (b_i - a_i)^n$ infinite measure ($\nu(\mathbb{R}^n) = \infty$)
A probability measure	$P(E) =$ probability E happens finite measure (P(sample space) $= 1$)

Weak Convergence

Let $\mathcal{P}(X) = \{$all finite measures on $X\}$.
$\neq$ power set

Topology of Weak Convergence

$\mathcal{O}_{\mathrm{WC}(X)}$ on $\mathcal{P}(X)$
$\mathcal{O}_{\mathrm{WC}(X)} =$ the coarsest topology making $(\lambda\mu.\ \int f \mathrm{d}\mu)$ continuous $\forall f \in \mathrm{C_b}(X)$.

$$\mathrm{C_b}(X) = \{f : X \to \mathbb{R},\ f \text{ is bounded continuous}\}$$

$$\begin{array}{rcc} (\lambda\mu.\int f \mathrm{d}\mu) : \mathcal{P}(X) & \to & \mathbb{R} \\ \uplus & & \uplus \\ \mu & \mapsto & \int f \mathrm{d}\mu \end{array}$$

Fact. X is separable metrizable $\Longrightarrow$ So is $\mathcal{P}(X)$
X is Polish $\Longrightarrow$ So is $\mathcal{P}(X)$

Weak Convergence

Weak Convergence

$$\mu_n \Rightarrow_{\mathrm{wc}} \mu \overset{\text{def}}{\iff} \mu_n \longrightarrow \mu \text{ in } \mathcal{P}(X)$$

$$\mu_n \Rightarrow_{\mathrm{wc}} \mu \iff \forall f \in \mathrm{C_b}(X).\ \int f \mathrm{d}\mu_n \longrightarrow \int f \mathrm{d}\mu$$

Ex. The central limit theorem

$X_n \cdots$ i.i.d. samples,
$P_n \cdots$ the distribution of normalized sample mean of $X_0, \ldots, X_n$,

Under appropriate conditions, $P_n \Rightarrow_{\mathrm{wc}} \mathrm{Normal}(0, 1)$

Prokhorov's Theorem

$\mathcal{P}_r(X) = \mathcal{P}(X) \cap \{\mu.\ \mu(X) \leq r\}$ for some $r < \infty$

Prokhorov's Theorem

X : a Polish space,
$\Gamma \subseteq \mathcal{P}_r(X)$

$\overline{\Gamma}$ is compact in $\mathcal{P}(X) \iff \Gamma$ is tight: i.e.,

$$\forall \varepsilon > 0.\ \exists K : \text{compact in } X, \text{ s.t.} \forall \mu \in \Gamma.\ \mu(X - K) \leq \varepsilon$$

Fact For a metrizable space X and $A \subseteq X$,

$$A \text{ is compact } \iff \forall \{x_n\}_{n\in\mathbb{N}} \subseteq A.\ \exists n_k.\ \exists x \in A. \text{ s.t. } x_{n_k} \longrightarrow x$$

Obtain a weak converging subsequence!

Corollary

If X is separable and metrizable, $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}_r(X)$,
and $\{\mu_n\}_{n\in\mathbb{N}}$ is tight.

Then, $\exists\{\mu_{n_k}\}_{k\in\mathbb{N}}$: subsequence and μ s.t. $\mu_{n_k} \Rightarrow_{\mathrm{wc}} \mu$.

Prokhorov's Theorem

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Used for

- Completeness of the Lévy-Prokhorov metric
- Central limit theorem
- Sanov's theorem in large deviation theory
- Existence of optimal coupling in transportation theory

depending on

- Riesz representation theorem
- Alaoglu's theorem

Summary

Formalization in Isabelle/HOL

- Prokhorov's theorem
- Riesz representation theorem (Tough! 2.1k– LOC)
- (A special case of) Alaoglu's theorem

Archive of formal proofs

- *The Lévy-Prokhorov Metric*, June 2024 (6.6K lines)
  - the special case of Alaoglu's theorem
  - Prokhorov's theorem
- *The Riesz representation theorem*, June 2024 (4.4K lines)

Prokhorov's Theorem

A key lemma for the proof of Prokhorov's Theorem

If X is a compact metric space, then $\mathcal{P}_r(X)$ is compact.

Idea: $\mathcal{P}_r(X) \cong \langle$a compact space$\rangle$

$$\begin{array}{ccc} T : \mathcal{P}_r(X) & \to & \mathbb{R}^{\mathrm{C}(X)} \cap \{\varphi.\ \varphi \text{ is positive linear } \wedge \varphi(1) \le r\} \quad (=: \Phi) \\ \Downarrow & & \Downarrow \\ \mu & \mapsto & \left(\lambda f.\ \int f \mathrm{d}\mu\right) \end{array}$$

Linearity $T(\mu)(f+g) = \int f + g \, \mathrm{d}\mu = \int f \mathrm{d}\mu + \int g \mathrm{d}\mu = T(\mu)(f) + T(\mu)(g)$

- Inverse function?
  $\Longrightarrow$ The Riesz representation theorem
- Compactness of Φ?
  $\Longrightarrow$ Alaoglu's theorem

Riesz Representation Theorem

The $\begin{cases} \text{Riesz} \\ \text{Riesz-Markov} \\ \text{Riesz-Markov-Kakutani} \end{cases}$ representation theorem.

Riesz Representation Theorem

X, a locally compact Hausdorff space
$\mathrm{C_C}(X)$, the set of continuous functions which have closed compact supports

$\varphi : \mathrm{C_C}(X) \to \mathbb{R}$, a positive linear functional
i.e., $\varphi(\alpha f + \beta g) = \alpha\varphi(f) + \beta\varphi(g)$ and $\varphi(f) \geq 0$ if $f \geq 0$.

Then, there exists $\mathcal{M} \supseteq \sigma[\mathcal{O}_X]$ and a unique measure μ on $\mathcal{M}$ s.t.

$$\forall f \in \mathrm{C_C}(X).\ \varphi(f) = \int f \mathrm{d}\mu \quad + 5 \text{ conditions}$$

Rudin's book

- 9 pages including lemmas (e.g. Urysohns' lemma)

Formal proof

- 2.1k+ lines

Set-Based Vector Space

$\varphi : \mathrm{C_C}(X) \to \mathbb{R}$, a positive linear functional

$\mathrm{C_C}(X)$: a vector space

Vector space in Isabelle/HOL

- Type-based (**class**) by [Hölzl+, ITP2013]
  - carrier sets must be *UNIV* (all elements of the type)

    Ex. Vector spaces on

    ☺ $\mathbb{C}$

    ☹ $\mathrm{C_C}(X)$

- Set-based (**locale**, **typedef**)
  - any carrier sets
  - has only basic definitions ([Lee, AFP2014])

My choice

- do not use vector space library
- write down conditions directly

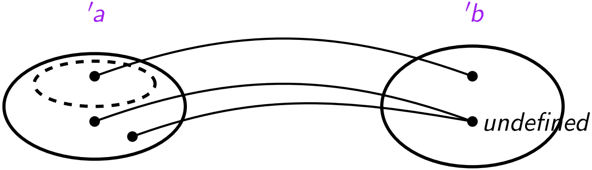
Riesz Representation Theorem in Isabelle/HOL

$\varphi : \mathrm{C_C}(X) \to \mathbb{R}$, a positive linear functional

definition *positive-linear-functional-on-CX* ::
$'a$ *topology* $\Rightarrow$ $(('a \Rightarrow 'b$ :: {*ring*, *order*, *topological-space*}$) \Rightarrow 'b) \Rightarrow$ *bool*

where *positive-linear-functional-on-CX X* $\varphi \equiv$
($\forall$ *f*. *continuous-map X euclidean f* $\longrightarrow$ *f has-compact-support-on X*
$\longrightarrow$ ($\forall$ *x*$\in$*topspace X*. *f x* $\geq$ *0*) $\longrightarrow$ φ ($\lambda x\in$*topspace X*. *f x*) $\geq$ *0*) $\wedge$
\+ *linearity*

- $'b$ :: {*ring*, *order*, *topological-space*}
  for *real* and *complex*
- $(\lambda x\in \textit{topspace } X.\ f\ x)\ y = \begin{cases} f\ y & \text{if } y \in \textit{topspace } X \\ \textit{undefined} & \text{o.w.} \end{cases}$



Alaoglu's Theorem

Y : a normed vector space
Y^* : the dual space
weak* topology on Y^* : the coarsest topology making all $(\lambda f.\, f(y)) : Y^* \to \mathbb{R}$ continuous

Alaoglu's Theorem

Let $B^* = \{\varphi \in Y^* \mid \|\varphi\| \leq r\}$, then B^* is compact in Y^* w.r.t. weak* topology.

Set-based vector spaces have neither dual space nor norm.

My choice Prove the spacial case for Prokhorov's theorem

Special Case of Alaoglu's Theorem

If X is compact, then $\mathbb{R}^{\mathrm{C}(X)} \cap \{\varphi.\, \varphi \text{ is positive linear } \wedge \varphi(1) \leq r\}$ is compact.

$\|\varphi\| = \varphi(1)$ if φ is positive linear.

Alaoglu's Theorem in Isabelle/HOL

Special Case of Alaoglu's Theorem

If X is compact, then $\mathbb{R}^{\mathrm{C}(X)} \cap \{\varphi.\, \varphi$ is positive linear $\wedge\, \varphi(1) \leq r\}$ is compact.

theorem *Alaoglu-theorem-real-functional*:
  fixes *X* :: *'a topology* **and** *r* :: *real*
  defines *prod-space* $\equiv \mathbb{R}^{\mathrm{C}(X)}$
  defines *B* $\equiv$ {*φ∈topspace prod-space*.
      φ (*λx∈topspace X*. *1*) $\leq$ *r* ∧ *positive-linear-functional-on-CX X φ*}
  assumes *compact-space X* **and** *topspace X* $\neq$ {}
  shows *compactin prod-space B*

Related Works

In Isabelle/HOL by Avigad et al. (2017)

A special case of Prokhorov's theorem for the central limit theorem

- The special case

  $\Longrightarrow$ a simpler proof

- General case

  $\Longrightarrow$ needs Riesz representation, Alaoglu's theorem

In Lean

`RieszMarkovKakutani.lean`

> *This file will prove different versions of the Riesz-Markov-Kakutani representation theorem. ...*

I could not find the final statements. It seems still ongoing.

Conclusion

Formalization in Isabelle/HOL

- Prokhorov's theorem
- The Riesz representation theorem
- A special case of Alaoglu's theorem

Reference

- Prokhorov's theorem

  Onno van Gaans, *Probability measures on metric spaces*,
  `https://www.math.leidenuniv.nl/~vangaans/jancol1.pdf`

- The Riesz representation theorem

  Rudin, Walter, *Real and Complex Analysis, 3rd Ed*, 1987

- Alaoglu's theorem

  Christopher E. Heil, *Alaoglu's Theorem*,
  `https://heil.math.gatech.edu/6338/summer08/section9f.pdf`